Complexity of the Continued Fractions of Some Subrecursive Real Numbers

Ivan Georgiev[1]

Prof. Assen Zlatarov University, Burgas, Bulgaria

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The classes we are interested in are the third ($\mathcal{E}^2$) and the fourth ($\mathcal{E}^3$) level of Grzegorczyk's hierarchy of $\mathcal{PR}$.

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Definition 2

The class $\mathcal{E}^2$ is the smallest class of total functions in $\mathbb{N}$, which contains the constant 0, the successor function $\lambda x.x+1$, the projections $\lambda x_1 \dots x_n.x_i (i, n \in \mathbb{N}, 1 \leq i \leq n)$, the addition function and the multiplication function and is closed under substitution and bounded primitive recursion.

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We consider two sequences p and q, defined by

- $p_{-1} = 1,\ p_0 = a_0,\ p_{n+1} = a_{n+1}p_n + p_{n-1},\ n = 0, 1, \ldots$
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It is true that $b_n = \frac{p_n}{q_n}$ for all $n \in \mathbb{N}$.

A generalization

In my master's thesis [1] the following theorem is proven

Theorem 3

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If ξ has continued fraction in $\mathcal{E}^2$, then ξ is $\mathcal{E}^2$-computable.

The proof utilizes a function r, defined by $r(n,t) = min(q_n, t+1)$ and the crucial step is to prove that $r \in \mathcal{E}^2$, which is done by the following representation (similar to the definitions of p and q):

$$r(0,t) = min(q_0, t+1), r(1,t) = min(q_1, t+1),$$

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A simple, but important observation, which allows an essential generalization of the theorem is the following: the number a_{n+2} in equality (1) can be changed to $min(a_{n+2}, t+1)$ without effect on its correctness. So, to conclude that $r \in \mathcal{E}^2$, it is sufficient to have

$$\lambda nt.min(a_{n+2}, t+1) \in \mathcal{E}^2 \quad (2)$$

(not the stronger $\lambda n.a_{n+2} \in \mathcal{E}^2$).

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We apply theorem 4 for $\xi_A = A(0,0) + \frac{1}{A(1,1)+\frac{1}{A(2,2)+\frac{1}{\ddots}}}$, where A is Ackermann's function.

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Hence, the number ξ_A is $\mathcal{E}^2$-computable, but its continued fraction is not primitive recursive, let alone in $\mathcal{E}^2$.

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We conclude that the converse of theorem 3 is not true.

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The number ξ is *$\mathcal{F}$ – irrational* if there exists unary function $v \in \mathcal{F}$, such that for all natural m and $n > 0$, $\left|\xi - \frac{m}{n}\right| > \frac{1}{v(n)}$.

Theorem 5 (Lehman)

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By a close scrutiny of the proof in [4] of this theorem we obtain the following

Theorem 6

If the number ξ is $\mathcal{E}^2$-computable and $\mathcal{E}^2$-irrational, then ξ has continued fraction in $\mathcal{E}^3$.

Applications

Finally, we will apply theorem 6 to obtain some interesting facts.

Definition 7

Let R be the set of all positive real numbers r, such that the inequality

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^r}$$

has at most finitely many solutions (p, q), where p and $q > 0$ are integers. The infimum of R is called the irrationality measure of ξ.

Lemma 8

If ξ has finite irrationality measure, then ξ is $\mathcal{E}^2$-irrational.

Applications

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Thank you for your attention!