Computing Real Functions with Rudimentary Operators

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Introduction

For natural $m \geq 1$, the set $\{a | a : \mathbb{N}^m \to \mathbb{N}\}$ will be denoted by $\mathcal{T}_m$. The union $\bigcup_{m\geq 1}^{\infty} \mathcal{T}^m$ will be denoted by $\mathcal{T}$.

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3. show that $\mathbb{MSO}$ is a proper subclass of $\mathbb{RO}$;
4. compare the computational power of these classes with respect to real functions by applying a general characterization theorem of D. Skordev in [1].

The class of functions $\mathcal{M}^2$

Let $a \in \mathcal{T}_{m+1}$. We define $b \in \mathcal{T}_{m+1}$ by

$$b(\overline{x}, y) = \begin{cases} z, & \text{if } z \le y, a(\overline{x}, z) = 0 \text{ and } \forall t < z[a(\overline{x}, t) \ne 0] \\ y + 1, & \text{if } \forall t \le y[a(\overline{x}, t) \ne 0] \end{cases}$$

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The functions $\lambda x_1 \dots x_n . x_m (1 \leq m \leq n)$, $\lambda x . x + 1$, $\lambda xy . x \dot{-} y$, $\lambda xy . xy$ and $\lambda xy . \lfloor \frac{x}{y+1} \rfloor$ will be called *the initial functions*.

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It is true that $\mathcal{M}^2 \subseteq \mathcal{E}^2$ and that $\mathcal{M}^2$ contains exactly the functions in $\mathcal{T}$, which are bounded by polynomial and Δ_0-definable.

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$$\Phi(\overline{f})(\overline{x}) = \Phi_0(\overline{f})(\Phi_1(\overline{f})(\overline{x}), \ldots, \Phi_k(\overline{f})(\overline{x}))$$
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Proposition

Let m, n be non-zero and Φ be an (n, m)-operator belonging to $\mathbb{MSO}$. There exists a natural number v with the property that for all $f_1, \ldots, f_n$ and $x_1, \ldots, x_m$ there exists a finite set A of at most v natural numbers, such that $\Phi(g_1, \ldots, g_n)(\overline{x}) = \Phi(f_1, \ldots, f_n)(\overline{x})$ whenever $g_l(t) = f_l(t)$ for all $l \in \{1, \ldots, n\}$ and $t \in A$.

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Proposition

For non-zero n, if Φ_0 is a rudimentary $(n, m+1)$-operator, then so is the operator Φ defined by $\Phi(\overline{f})(\overline{x}, y) = \max_{z \leq y} \Phi_0(\overline{f})(\overline{x}, z)$.

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Proposition

$\mathbb{MSO} \subseteq \mathbb{RO}$ *and* $\mathbb{MSO} \neq \mathbb{RO}$.

The $(1,1)$-operator Φ, defined by $\Phi(f)(x) = \max_{y \leq x} f(y)$ is rudimentary, but not $\mathcal{M}^2$-substitutional.

The Uniformity Theorem

Definition

Let m, n be non-zero and Φ be an (n, m)-operator. We say that the $(1, m)$-operator Ω *uniformizes* Φ if for all $\overline{x}, f$ and all $g_1, \ldots, g_n, h_1, \ldots, h_n$ dominated by f, if $g_1(t) = h_1(t), \ldots, g_n(t) = h_n(t)$ for all $t \leq \Omega(f)(\overline{x})$, then $\Phi(\overline{g})(\overline{x}) = \Phi(\overline{h})(\overline{x})$.

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Theorem

Let m, n be non-zero and Φ be an $\mathcal{M}^2$-substitutional (n,m)-operator. There exists an $\mathcal{M}^2$-substitutional $(1,m)$-operator Ω which uniformizes Φ.

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The notion of acceptable pair I

Definition (Skordev, [1])

Let $\mathcal{F} \subseteq \mathcal{T}$ and $\mathbb{OP} \subseteq \mathbb{O}$. The pair $(\mathcal{F}, \mathbb{OP})$ will be called *acceptable*, if the following conditions hold:

- The initial functions belong to $\mathcal{F}$.
- $\mathcal{F}$ is closed under substitution.
- For non-zero n, the $(n,1)$-operator Φ defined by $\Phi(\bar{f})(x) = x$ belongs to $\mathbb{OP}$.
- For non-zero n and $k \in \{1, \ldots, n\}$, if Φ_0 is an $(n,1)$-operator which belongs to $\mathbb{OP}$, then the $(n,1)$-operator Φ defined by $\Phi(\bar{f})(x) = f_k(\Phi_0(\bar{f})(x))$ also belongs to $\mathbb{OP}$.
- For non-zero n, k and $a \in \mathcal{T}_k \cap \mathcal{F}$, if $\Phi_1, \ldots, \Phi_k$ are $(n,1)$-operators which belong to $\mathbb{OP}$, then so is the operator Φ defined by

$$\Phi(\bar{f})(x) = a(\Phi_1(\bar{f})(x), \ldots, \Phi_k(\bar{f})(x)).$$

The notion of acceptable pair II

Definition (continued)

- For non-zero n, if $a_1, \ldots, a_n \in \mathcal{T}_{l+1} \cap \mathcal{F}$ and $\Phi \in \mathbb{OP}$ is an $(n,1)$-operator, then the function

$$\lambda s_1 \ldots s_l x.\Phi(\lambda t.a_1(s_1, \ldots, s_l, t), \ldots, \lambda t.a_n(s_1, \ldots, s_l, t))(x)$$

  belongs to $\mathcal{F}$.
- For non-zero n and for every $(n,1)$-operator $\Phi \in \mathbb{OP}$, there exists an $(n,1)$-operator $\Omega \in \mathbb{OP}$ which uniformizes Φ.

Theorem

The pairs $(\mathcal{M}^2, \mathbb{MSO})$ *and* $(\mathcal{M}^2, \mathbb{RO})$ *are acceptable.*

Uniform computability

A triple $(f, g, h) \in \mathcal{T}_1^3$ *names* a real number ξ, if for all t

$$\left| \frac{f(t) - g(t)}{h(t) + 1} - \xi \right| < \frac{1}{t + 1}.$$

Definition (Skordev, [1])

Let $\mathbb{OP}$ be a class of operators ($\mathbb{OP} \subseteq \mathbb{O}$). For non-zero l, a real function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}^l$, will be called *uniformly $\mathbb{OP}$-computable*, if there exist $(3l, 1)$-operators $F, G, H \in \mathbb{OP}$, such that for all l-tuples $(\xi_1, \ldots, \xi_l) \in D$ and l triples $(f_1, g_1, h_1), \ldots, (f_l, g_l, h_l)$ naming $\xi_1, \ldots, \xi_l$, respectively, the triple

$$\begin{gathered}(F(f_1, g_1, h_1, \ldots, f_l, g_l, h_l),\\ G(f_1, g_1, h_1, \ldots, f_l, g_l, h_l),\\ H(f_1, g_1, h_1, \ldots, f_l, g_l, h_l))\end{gathered}$$

names $\theta(\xi_1, \ldots, \xi_l)$.

Uniform TZ-style computability

Definition (Tent and Ziegler, [5])

Let $\mathcal{F}$ be a subclass of $\mathcal{T}$. A real function $\theta : D \to \mathbb{R}$, where $D \subseteq \mathbb{R}\mathbb{O}^l$, $l > 0$, will be called *uniformly TZ-style $\mathcal{F}$-computable*, if there exist $d \in \mathcal{T}_1 \cap \mathcal{F}$ and $a, b, c \in \mathcal{T}_{3l+1} \cap \mathcal{F}$, such that for all $(\xi_1, \dots, \xi_l) \in D$ and $x_1, y_1, z_1, \dots, x_l, y_l, z_l, s$ the inequalities

$$|\xi_k| \leq s + 1, \left| \frac{x_k - y_k}{z_k + 1} - \xi_k \right| < \frac{1}{d(s) + 1}$$

for $k \in \{1, \dots, l\}$, imply that the numbers

$$x = a(x_1, y_1, z_1, \dots, x_l, y_l, z_l, s), y = b(x_1, y_1, z_1, \dots, x_l, y_l, z_l, s),$$

$$z = c(x_1, y_1, z_1, \dots, x_l, y_l, z_l, s)$$

satisfy the inequality

$$\left| \frac{x - y}{z + 1} - \theta(\xi_1, \dots, \xi_l) \right| < \frac{1}{s + 1}.$$

The main result

Theorem

The following three conditions are equivalent for a real function $\theta : D \to \mathbb{R}$*, where* $D \subseteq \mathbb{R}^l, l > 0$.

1. θ *is uniformly* $\mathbb{RO}$*-computable.*
2. θ *is uniformly TZ-style* $\mathcal{M}^2$*-computable.*
3. θ *is uniformly* $\mathbb{MSO}$*-computable.*

Proof.

We apply the characterization theorem of Skordev in [1], which states that for an acceptable pair $(\mathcal{F}, \mathbb{OP})$,
θ is uniformly $\mathbb{OP}$-computable if and only if
θ is uniformly TZ-style $\mathcal{F}$-computable. □

References

- Skordev, D., On some computability notions for real functions. (submitted for publication in Computability Journal)
- Skordev, D., Some subrecursive versions of Grzegorczyk's Uniformity Theorem. Math. Logic Quart., 50, 520–524 (2004)
- Skordev, D., Uniform computability of real functions. In: 120 Years Faculty of Mathematics and Informatics, St. Kliment Ohridski University of Sofia, Proceedings, St. Kliment Ohridski Press, Sofia, 179–185 (2011)
- Skordev, D., Weiermann, A., Georgiev, I., $\mathcal{M}^2$-computable real numbers. J. Logic and Computation (Advance Access published September 21, 2010), doi:10.1093/logcom/exq050
- Tent, K., Ziegler, M., Computable functions of reals. Münster J. Math., 3, 43–66 (2010)

Thank you for your attention!